Radner Equilibrium: Definition and Equivalence with Arrow-Debreu Equilibrium, Asset Markets and No Arbitrage

Econ 3030 Fall 2025

Lecture 26

Outline

1. Sequential Trade and Arrow Securities
2. Radner Equilibrium
3. Equivalence between Arrow-Debreu and Radner Equilibria
4. Asset Markets and No Arbitrage

Arrow-Debreu Equilibrium in the One Good Timeless Exchange Economy

- One commodity, no firms, one future period, and no consumption today;
  - dated events are $(1, \{s\})$, so we label everything by the realized state $s \in S$.
- A consumption bundle is an S-dimensional vector $\mathbf{x}_i \in X_i \subset \mathbb{R}_+^S$ and

$$\mathbf{x}_i \succsim \mathbf{y}_i \quad \text{if and only if} \quad \sum_{s=1}^{S} \pi_{si} u_i(x_{si}) \geq \sum_{s=1}^{S} \pi_{si} u_i(y_{si})$$

Definition

$\mathbf{x}^* \in \mathbb{R}_+^{S \times I}$ and $\mathbf{p}^* \in \mathbb{R}_+^S$ are an Arrow-Debreu equilibrium if

1. For every i, $\mathbf{x}_i^* \in \mathbb{R}_+^S$ solves: $\max_{\mathbf{x}_i \in \mathbb{R}_+^S} \sum_{s=1}^{S} \pi_{si} u_i(x_{si})$ subject to $\sum_{s=1}^{S} p_s^* x_{si} \leq \sum_{i=1}^{I} p_s^* \omega_{si}$
2. Markets clear: $\sum_{i=1}^{I} x_{si}^* \leq \sum_{i=1}^{I} \omega_{si} \qquad \text{for } s = 1, \ldots, S$

Timing of Trades in Arrow-Debreu

Remark

In an Arrow-Debreu economy, all decisions are made at date 0.

- Individuals exchange promises to deliver and receive quantities of the goods according to the realized dated-event.
  - They trade now for all future deliveries.
- As time and uncertainty unfold, these promises are carried through exactly as planned, without changes.
- What if tomorrow there were also markets for the L goods?
- Is there an incentive to trade in these spot markets?

Spot Markets Do Not Matter in Arrow-Debreu

Fact

Given an Arrow-Debreu equilibrium, there are no incentives to trade in spot markets.

Proof.

Suppose not: consumers can find a mutually beneficial trade in some dated-event.

- Thus, in that dated-event there exist a feasible allocation that consumers prefer to the equilibrium bundle, with at least one consumer preference being strict.
  - consumers trade only if they can do better.
- This new allocation is feasible, and Pareto dominates the Arrow-Debreu equilibrium.
- But this is impossible, because the First Welfare Theorem holds. □

Spot and Forward Markets

Remark

Most real world markets are spot, not forward.

Objective

Reconcile this observation with a general equilibrium model with uncertainty similar to Arrow-Debreu.

Idea

- The main role of state contingent commodities is to allow welfare transfers across states...
- If one can have achieve that same objective with just a few dated-event contingent commodities,
- then only some forward markets are needed, and most trade takes place in spot markets.

Radner Equilibrium

- Individuals make decisions now and in the future
  - only one commodity is traded now and in the future
    - this commodity can be used to transfer wealth across states,
  - all other commodities are traded only in markets that open after uncertainty is resolved.
- Today's decisions depend on forecasts about the future: trade is sequential, and expectations are crucial.
- This equilibrium concept is called Radner equilibrium.

Main Idea

- Replace forward markets with expectations about future spot markets.
- Given these, consumers decide about welfare transfers across states at date 0.
- Afterwards, consumers trade in markets for physical goods.

Main Result: Radner and Arrow-Debreu Are Equivalent

If expectations are correct, and individuals have some effective way to transfer wealth across states, a Radner equilibrium is equivalent to an Arrow-Debreu equilibrium: having fewer markets does not matter.

Sequential Trading and Arrow Securities

Date 0

Only one physical good is used as a state-contingent commodity: money.
Amounts of money are traded for delivery if and only if a particular state occurs.

- These are called Arrow securities.

Date 1

Some state occurs, and markets open where commodities trade at current prices.

- Individuals trade decisions depend on (i) current prices, and (ii) their income;
  - income depends on their trades in the state-contingent commodity that corresponds to the realized state.

Expectations Are Crucial

Date 0 trades reflect what consumers think will happen in the future markets.

- To decide how much to trade of the state-contingent commodity individuals make consumption plans for each possible state; these plans depend on their forecasts of the future spot prices.

Sequential Trading and Arrow Securities in an Exchange Economy

Notation

- $\mathbf{z}_i = (z_{1i}, .., z_{Si}) \in \mathbb{R}^S$ denotes i's trades in the state-contingent commodity;
  - this *Arrow security* specifies amounts to be delivered, or received, of commodity 1 in each state.
    - **note**: these trades can be negative or positive.
- $\mathbf{q} = (q_1, .., q_S) \in \mathbb{R}^S_+$ denotes the prices of the Arrow security.
- $\mathbf{x}_i = (\mathbf{x}_{1i}, .., \mathbf{x}_{Si}) \in \mathbb{R}^{L \times S}_+$ denotes i's consumption plans vector;
  - $\mathbf{x}_{si} = (x_{1si}, .., x_{Lsi}) \in \mathbb{R}^L_+$ denotes i's expected consumption in state s;
    - **note**: $\mathbf{x}_{1i} = (x_{11i}, .., x_{1Si}) \in \mathbb{R}^S_+$ represents expected trade in commodity 1, the commodity for which there are also date 0 markets.
- $\mathbf{p} = (\mathbf{p}_1, .., \mathbf{p}_S) \in \mathbb{R}^{L \times S}_+$ is the vector of expected prices;
  - $\mathbf{p}_s \in \mathbb{R}^L_+$ is the expected price vector for the L goods in state s.

- **Remark**: $\mathbf{x}_i$ and $\mathbf{p}$ are expected: they represent planned choices and prices.
- Consumers make choices and plans given current and expected prices.
  - There is no date 0 consumption (for simplicity).

Optimization

Given current prices ($\mathbf{q}$) and expected prices ($\mathbf{p}$), each individual maximizes utility.

- A consumption plan includes current trades in the state-contingent commodity ($\mathbf{z}_i$) and expected purchases in each possible state ($\mathbf{x}_i$).

Consumers' Choices at time 0

At date 0, consumer i solves the following maximization problem

$$\max_{\mathbf{z}_i \in \mathbb{R}^S, \mathbf{x} \in \mathbb{R}_+^{L \times S}} U_i(\mathbf{x})$$

subject to $\underbrace{\mathbf{q} \cdot \mathbf{z}_i \leq 0}_{\text{budget constraint at time 0}}$ and $\underbrace{\mathbf{p}_s \cdot \mathbf{x}_{si} \leq \mathbf{p}_s \cdot \boldsymbol{\omega}_{si} + p_{1s} z_{si} \quad \text{for each } s}_{\text{expected budget constraints at time 1}}$

- There will be S budget constraints at date 1: from date 0 perspective, these constraints are "in expectation": $\mathbf{x}_{si}$ is expected consumption; $\mathbf{p}_s$ are expected prices.
- **Remark**: at date 1 good 1 enters in three parts:
  - buy the desired amount for immediate consumption (x_{1si}),
  - sell the endowment (ω_{1si}), and
  - sell the realized state-contingent "outcome" of date 0 trades (z_{si}).

Time 0 Budget Constraint

Date 0 budget constraint

$$\mathbf{q} \cdot \mathbf{z}_i = \sum_{s=1}^{S} q_s z_{si} \leq 0$$

- The individual has no income at time 0, hence her trades cannot have positive cost.
  - Since the price vector is non-negative, if she promises to buy good 1 in state s ($z_{si} > 0$), she must also promise to sell it in some other state t ($z_{ti} < 0$).
  - She cannot promise to spend more than she makes
    - one could add a positive date 0 endowment (and date zero consumption), without affecting this logic.
- The time 0 budget constraint is homogeneous of degree zero in prices.
  - Therefore, we can normalize time 0 prices.

Time 1 Expected Budget Constraints

Date 1 expected budget constraints

$$\sum_{l=1}^{L} p_{ls} x_{lsi} \leq \sum_{l=1}^{L} p_{ls} \omega_{lsi} + p_{1s} z_{si} \qquad \text{for each } s = 1, \ldots, S$$

- In each state, the individual expects to trade in the markets that correspond to that state; her expected wealth reflects the outcome of time 0 trades.
- We can normalize by assuming that $p_{s1} = 1$ for each s.
  - Thus, Arrow security promises to receive/deliver units of money (good 1).
- Since there are no "sign" restrictions on $\mathbf{z}_i$, we may have $z_{si} < -\omega_{1si}$ in some state s:
  - i promises to deliver more than she will have, and her income must compensate.
    - the ability to do this is limited because consumption cannot be negative.

- Arrow securities allow wealth transfers across states: at date 0, a consumer can buy a state s dollar and pay for it with a state t dollar.
  - If state s occurs, she uses the extra dollar to buy commodities;
  - if state t occurs, she has one less dollar to buy commodities.

Radner Equilibrium

In equilibrium, everything happens exactly as planned.

Definition

A Radner equilibrium is composed by state-contingent commodities prices $\mathbf{q}^* \in \mathbb{R}^S$, trades $\mathbf{z}_i^* \in \mathbb{R}^S$, spot prices $\mathbf{p}_s^* \in \mathbb{R}^L$ for each s, and consumption $\mathbf{x}_i^* \in \mathbb{R}^{L\times S}$ for each i such that:

1. for each individual, $\mathbf{z}_i^*$ and $\mathbf{x}_i^*$ solve

$$\max_{\mathbf{z}_i \in \mathbb{R}^S, \mathbf{x}_{si} \in \mathbb{R}^L_+} \sum_{s=1}^S \pi_{si} u_{si}(\mathbf{x}_{si}) \quad \text{subject to} \quad \begin{array}{l} (i) : \sum_{s=1}^S q_s^* z_{si} \le 0 \\ (ii) : \mathbf{p}_s^* \cdot \mathbf{x}_{si} \le \mathbf{p}_s^* \cdot \boldsymbol{\omega}_{si} + p_{1s}^* z_{si} \quad \text{for all } s \end{array}$$

2. all markets clear:

$$\sum_{i=1}^I \mathbf{z}_{si}^* \le 0 \qquad \text{and} \qquad \sum_{i=1}^I \mathbf{x}_{si}^* \le \sum_{i=1}^I \boldsymbol{\omega}_{si} \qquad \text{for all } s$$

- Markets clear at consumption plans that maximize individuals' utility, given current and expected prices.

Radner and Arrow-Debreu Are Equivalent

- The timing of decisions is unimportant.

Proposition (Equivalence of Arrow-Debreu and Radner equilibria)

1. *Let the allocation* $\mathbf{x}^* \in \mathbb{R}^{L\times S\times I}$ *and the prices* $\mathbf{p}^* \in \mathbb{R}^{L\times S}_{++}$ *constitute an Arrow-Debreu equilibrium. Then, there are prices* $\mathbf{q}^* \in \mathbb{R}^{S}_{++}$ *and state-contingent commodity trades* $\mathbf{z}^* = (\mathbf{z}_1^*, .., \mathbf{z}_I^*) \in \mathbb{R}^{S\times I}$ *such that:* $\mathbf{z}^*$, $\mathbf{q}^*$, $\mathbf{x}^*$, *and* $\mathbf{p}_s^*$ *for each* s *form a Radner equilibrium.*
2. *Let* $\mathbf{x}^* \in \mathbb{R}^{L\times S\times I}$, $\mathbf{z}^* \in \mathbb{R}^{S\times I}$, $\mathbf{q}^* \in \mathbb{R}^{S}_{++}$ *and* $\mathbf{p}_s^* \in \mathbb{R}^{L}_{++}$ *for all* s *form a Radner equilibrium. Then, there are* S *strictly positive numbers* $\mu_1, .., \mu_S$ *such that the allocation* $\mathbf{x}^*$ *and the state-contingent commodities price vector* $(\mu_1\mathbf{p}_1^*, .., \mu_S\mathbf{p}_S^*) \in \mathbb{R}^{L\times S}_{++}$ *form an Arrow-Debreu equilibrium.*

1. An Arrow-Debreu equilibrium can be made into a Radner equilibrium by appropriately choosing the state contingent commodity trades and prices.
2. A Radner equilibrium can be made into an Arrow Debreu equilibrium by appropriately modifies spot prices into state-contingent prices.

- The proof only needs to show that the two budget sets are the same.

The Importance of Arrow Securities

The propositon says that Arrow-Debreu equilibrium and Radner equilibrium are equivalent, provided there are enough Arrow securities.

Arrow-Debreu and Radner are equivalent

- An Arrow-Debreu equilibrium can be made into a Radner equilbrium.
- A Radner equilibrium can be made into an Arrow-Debreu equilibrium.

- Then the timing of trades is not that important.
  - Equilibrium outcomes are the same if we imagine all trade taking place at time 0, or if we imagine that Arrow securities are traded at that time and the other trades occur in spot markets.
- Having all the state-contingent commodities trades is not that important.
  - Equilibrium outcomes are the same if we imagine a full set of state-contingent commodities, or if we only imagine one Arrow security per state.

Asset Markets in General Equilibrium

- Next, we model financial assets (rather than state contingent commodities).
- A unit of an asset gives the holder the right to receive some future payment.

- An asset is a title to receive pre-specified amounts of good 1 at date 1.
- An asset is completely characterized by its return vector $\mathbf{r}_k = (r_{k1}, .., r_{kS}) \in \mathbb{R}^S$
  - r_{ks} is the dividend paid to the holder of a unit of r_k if and only if state s occurs.

Definition

The asset return matrix R is an $S \times K$ matrix whose kth column is the return vector of asset k. That is:

$$R = \begin{bmatrix} r_{11} & .. & r_{k1} & .. & r_{K1} \\ .. & .. & .. & .. & .. \\ r_{1s} & .. & r_{ks} & .. & r_{Ks} \\ .. & .. & .. & .. & .. \\ r_{1S} & .. & r_{kS} & .. & r_{KS} \end{bmatrix}$$

A row indicates the returns of all assets in one particular state.

Assets: Examples

Example

An asset that delivers one unit of good 1 in all states:

$$\mathbf{r} = (1, .., 1)$$

If there is only one good, $L = 1$, this the **risk-free** (or safe) asset.

- Why is this not safe with many goods?
  - Because the price of good 1 can change from state to state.
    - An asset matters insofar as it can be transformed into consumption goods; the rate at which one can do this depends on relative prices.

Example

An asset that delivers one unit of good 1 in one state, and zero otherwise:

$$\mathbf{r} = (0, .., 1, .., 0)$$

This is an Arrow security.

Budget Constraints with Asset Markets

Given an asset matrx R, one can define prices and holdings of each asset.

- $\mathbf{q} = (q_1, .., q_K) \in \mathbb{R}^K$ are the asset prices, where q_k is the price of asset k.
- $\mathbf{z}_i = (z_{1i}, .., z_{Ki}) \in \mathbb{R}^K$ are consumer i's holdings of each asset.
  - This is called a portfolio: it shows how many units of each asset i owns.

Assets are traded at time 0, while returns are realized at time 1. At that time, agents decide how much to consume and they trade in spot markets.

- As in Radner, i's budget constraints are

$$\underbrace{\sum_{k=1}^{K} q_k z_{ki} \leq 0}_{\text{time } 0} \qquad \text{and} \qquad \underbrace{\mathbf{p}_s \cdot \mathbf{x}_{si} \leq \mathbf{p}_s \cdot \boldsymbol{\omega}_{si} + \sum_{k=1}^{K} p_{1s} z_{ki} r_{sk} \qquad \text{for each } s}_{\text{time } 1}$$

- Income at time 1 is given by the value of endowment plus the income one obtains by selling the returns of the assets one owns.
  - As usual, one can normalize the spot price of good one to be 1.

Assets vs State-contingent Commodity in Radner

Question

What is the difference between these budget constraints

$$\sum_{k=1}^{K} q_k z_{ki} \leq 0 \qquad \text{and} \qquad \mathbf{p}_s \cdot \mathbf{x}_{si} \leq \mathbf{p}_s \cdot \boldsymbol{\omega}_{si} + \sum_{k=1}^{K} p_{1s} z_{ki} r_{sk} \qquad \text{for each } s$$

and the ones in a Radner equilibrium?

- Here, assets' dividends are given; one focuses only on the portfolio choice $\mathbf{z}_i$.
- In Radner, the dividends are 'constructed' by the consumers' choice of trades in the state-contingent commodity.
- Formally, in Radner one implicitly assumes S different assets, each with returns $r_s = 1$ in state s and zero otherwise.
- If $S = K$, we can write Radner using the z and r above:

$$z_s^{Radner} = \sum_{s=1}^{S} z_s r_s .$$

Budget Set

What do budget sets look like?

- Normalizing $p_{1s} = 1$, one can write i's budget set as

$$B_i(\mathbf{p}, \mathbf{q}, R) = \left\{ \mathbf{x}_i \in \mathbb{R}^{LS}_{+} : \begin{array}{c} \text{there is } \mathbf{z}_i \in \mathbb{R}^K \\ \text{such that} \end{array} \quad \begin{array}{c} \mathbf{q} \cdot \mathbf{z}_i \leq 0 \\ \text{and} \\ \mathbf{p} \cdot (\mathbf{x}_i - \boldsymbol{\omega}_i) \leq R\mathbf{z}_i \end{array} \right\}$$

- The second part of the budget set defines S inequalities

$$\begin{array}{ccc} \mathbf{p}_1 \cdot (\mathbf{x}_{1i} - \boldsymbol{\omega}_{1i}) & \leq & z_{1i} r_{11} + \ldots + z_{ki} r_{k1} + \ldots + z_{Ki} r_{K1} \\ \ldots & & \ldots \\ \mathbf{p}_s \cdot (\mathbf{x}_{si} - \boldsymbol{\omega}_{si}) & \leq & z_{1i} r_{1s} + \ldots + z_{ki} r_{ks} + \ldots + z_{Ki} r_{Ks} \\ \ldots & & \ldots \\ \mathbf{p}_S \cdot (\mathbf{x}_{Si} - \boldsymbol{\omega}_{Si}) & \leq & z_{1i} r_{1S} + \ldots + z_{ki} r_{kS} + \ldots + z_{Ki} r_{KS} \end{array}$$

  where each $\mathbf{p}_s \in \mathbb{R}^L$
- $R\mathbf{z}_i$ is the consumer's 'financial income' from choosing the portfolio $\mathbf{z}_i$.
- Given R, each value of $R\mathbf{z}_i$ determines a possible income available to the consumer.
  - Different $\mathbf{z}_i$s change the consumer's future income across states.

Radner Equilibrium with Asset Markets

Definition

A Radner equilibrium with assets is given by asset prices $\mathbf{q}^* \in \mathbb{R}^K$, prices $\mathbf{p}_s^* \in \mathbb{R}^L$ in each state s, portfolios $\mathbf{z}_i^* \in \mathbb{R}^K$, and consumption $\mathbf{x}_{si}^* \in \mathbb{R}^L$ for each s such that:

1. for each i, $\mathbf{z}_i^*$ and $\mathbf{x}_i^*$ solve
$$\max_{\mathbf{z}_i \in \mathbb{R}^K, \mathbf{x}_i \in \mathbb{R}_+^{S \times L}} U_i(\mathbf{x}_i)$$
subject to
$$\sum_{k=1}^{K} q_k^* z_{ki} \leq 0 \qquad \text{and} \qquad \mathbf{p}_s^* \cdot \mathbf{x}_{si} \leq \mathbf{p}_s^* \cdot \boldsymbol{\omega}_{si} + \sum_{k=1}^{K} p_{1s}^* z_{ki} r_{sk}$$

2. all markets clear; that is:
$$\sum_{i=1}^{I} z_{ki}^* \leq 0 \qquad \text{and} \qquad \sum_{i=1}^{I} \mathbf{x}_{si}^* \leq \sum_{i=1}^{I} \boldsymbol{\omega}_{si} \qquad \text{for all } s \text{ and } k$$

- The definition is familiar, but there are two new objects: the optimal portfolio and the equilibrium assets' prices.
- What can one say about them? What do we know about $\mathbf{q}^*$?

Complete Markets

Definition

An asset structure R is complete if $rank\ R = S$.

Example

There are S different Arrow securities; then

$$R = \begin{bmatrix} 1 & 0 & .. & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & 1 \end{bmatrix}$$

which is the identity matrix and therefore has rank S.

Example

There are three states, so $S = 3$, and $R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ which has full rank.

Arbitrage

Arbitrage

An abitrage opportunity is the possibility to make strictly positive profits at no risk.

- Under what conditions such opportunities do not exist?

Definition

Asset prices $\hat{\mathbf{q}} \in \mathbb{R}^K$ satisfy absence of arbitrage if there is no $\mathbf{z} \in \mathbb{R}^K$ such that

$$\hat{\mathbf{q}} \cdot \mathbf{z} \leq 0, \qquad R\mathbf{z} \geq 0 \qquad \text{and} \qquad R\mathbf{z} \neq 0.$$

- $\hat{\mathbf{q}}$ are sometimes called no-arbitrage prices.

- This can be interpreted as

$$\underbrace{\hat{\mathbf{q}} \cdot \mathbf{z} \leq 0}_{\text{affordable portfolio}} \quad , \qquad \underbrace{R\mathbf{z} \geq 0 \qquad \text{and} \qquad R\mathbf{z} \neq 0}_{\text{strictly positive profits for sure}}$$

- No arbitrage means there does not exists an affordable portfolio that yields non-negative returns in all states and strictly positive returns in some state.
- Preferences are nowhere to be seen in this definition.

Arbitrage and Asset Prices

- No arbitrage imposes restrictions on prices.

No arbitrage and prices

Let

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{so that} \qquad Rz = \begin{bmatrix} z_1 + z_2 + z_3 \\ z_1 + z_2 \\ z_1 \end{bmatrix}$$

- Here, no arbitrage implies that if $z_1 > 0$ then $q_1 > 0$. Why?
  - Can always pick z_2 and z_3 equal to zero.

- This result generalizes beyond the example.

No Arbitrage and Equilibrium Asset Prices

Theorem

Assume $r_k \geq 0$ and $r_k \neq 0$ for all k (return vectors are nonnegative and non zero), and suppose preferences are strictly monotone.
Then, any $\mathbf{q}^* \in \mathbb{R}^K$ that is part of a Radner equilibrium satisfies no-arbitrage.

- Under simple restrictions on assets and preferences, equilibrium implies absence of arbitrage.
- Intuitively, arbitrage implies some money is left on the table.
- Given the efficiency properties of equilibirum, this should not happen.
- This is similar in spirit to the First Welfare Theorem.
- Prove this as an exercise.

Equilibrium and Linear Asset Prices

Theorem (Martingale Pricing)

Assume $r_k \geq 0$ and $r_k \neq 0$ for all k (return vectors are nonnegative and non zero), and suppose preferences are strictly monotone. Then, for any $\mathbf{q}^* \in \mathbb{R}^K$ that is part of a Radner equilibrium, we can find non-negative numbers $\mu_s \geq 0$ which satisfy

$$q_k^* = \sum_{s=1}^{S} \mu_s r_{sk} \text{ for all } k$$

What does this equality imply?

- Prices are a linear combination of dividends: $[\mathbf{q}^*]^T = \boldsymbol{\mu} \cdot R$
- Suppose asset k pays a constant amount in all states: $r_{sk} = c$ for all s.
- Normalize the price of this asset to be c, so that $c = \sum_{s=1}^{S} \mu_s c$;then $1 = \sum_{s=1}^{S} \mu_s$
- Thus, equilibrium asset prices must equal an expected value of their returns.
- The 'probability' used to compute the expected value is the same for all assets.

No-Arbitrage Implies Linear Asset Prices

- Since we have already shown that equilibrium implies no-arbitrage, to prove the Martingale Pricing Theorem we only need to prove the following lemma.

Lemma

Suppose $r_k \geq 0$ and $r_k \neq 0$ for all k (return vectors are nonnegative and non zero). If asset prices $\mathbf{q} \in \mathbb{R}^K$ satisfy the no-arbitrage condition, then there exists a vector $\boldsymbol{\mu} \in \mathbb{R}^S_+$ such that

$$\mathbf{q}^T = \boldsymbol{\mu} \cdot R$$

- The proof follows from convexity of the set of no arbitrage portfolios.
  - The main step uses the separating hyperplane theorem, and should remind you of earlier proofs:
    - Separate the positive orthant from the set of incomes that can be obtained given a no-arbitrage (equilibirum) price vector.
- This is also left as an exercise.
- Notice that all we need for linear pricing is no-arbitrage.
- There are no restrictions on preferences for the Lemma!
- The connection with equilibrium is made by adding strictly monotone preferences.

Linear Pricing at Work

Implications of Linear Pricing

Suppose there is an asset that delivers one unit of good 1 in all states:

$$\mathbf{r}_1 = (1, .., 1)$$

- Let $\mathbf{q}$ be a no-arbitrage price vector and normalize $q_1 = 1$.
- If $\boldsymbol{\mu}$ satisfies $\mathbf{q}^T = \boldsymbol{\mu} \cdot R$, we have

$$\boldsymbol{\mu} = (\mu_1, .., \mu_S) \geq 0 \qquad \text{and} \qquad \sum_{s=1}^{S} \mu_s = \boldsymbol{\mu} \cdot r_1 = q_1 = 1$$

- Hence, for all assets different from asset 1,

$$q_k = \sum_{s=1}^{S} \mu_s r_{sk} \geq \min_s r_{sk} \qquad \text{and} \qquad q_k = \sum_{s=1}^{S} \mu_s r_{sk} \leq \max_s r_{sk}$$

- Intuitively, prices must be between the lowest and highest dividend.

Next Class

- Talk about the exam: it will be next Monday at 9:30 in this room and last 3 hours.
- Have some fun with Luca's research (this material will not be on the final).
- Material from today will also not be on the final.